

# A Deterministic Linear Quadratic Time-Inconsistent Optimal Control Problem\*

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## Abstract

A time-inconsistent optimal control problem is formulated and studied for a controlled linear ordinary differential equation with quadratic cost functional. A notion of equilibrium control is introduced, which can be regarded as a time-consistent solution to the original time-inconsistent problem. Under certain conditions, we constructively prove the existence of such an equilibrium control which is represented via a forward ordinary differential equation coupled with a backward Riccati–Volterra integral equation. Our constructive approach is based on the introduction of a family of  $N$ -person non-cooperative differential games.

**Keywords.** Time-inconsistency, linear-quadratic optimal control problem, equilibrium control, multi-level hierarchical differential games, backward Riccati–Volterra integral equation.

**AMS Mathematics subject classification.** 49L20, 49N10, 49N70, 91A23.

## 1 Introduction — Time-Consistency Issue.

We begin with a classical optimal control problem for an ordinary differential equation (ODE, for short). Let  $T > 0$ . For any *initial pair*  $(t, x) \in [0, T] \times \mathbb{R}^n$ , consider the following controlled ODE:

$$\begin{cases} \dot{X}(s) = b(s, X(s), u(s)), & s \in [t, T], \\ X(t) = x, \end{cases} \quad (1.1)$$

where  $b : [0, T] \times \mathbb{R}^n \times U \rightarrow \mathbb{R}^n$  is a given map,  $u(\cdot)$ , a function valued in some metric space  $U$ , is called a *control*, and  $X(\cdot)$  is called the *state trajectory*. We denote

$$\mathcal{U}[t, s] = \left\{ u : [t, s] \rightarrow U \mid u(\cdot) \text{ is measurable} \right\}, \quad \forall 0 \leq t \leq s \leq T. \quad (1.2)$$

Under some mild conditions, for any *initial pair*  $(t, x) \in [0, T] \times \mathbb{R}^n$ , and  $u(\cdot) \in \mathcal{U}[t, T]$ , (1.1) admits a unique solution  $X(\cdot) \equiv X(\cdot; t, x, u(\cdot))$ . Then we can introduce the following cost functional which measures the performance of the control  $u(\cdot)$ :

$$J(t, x; u(\cdot)) = \int_t^T g(s, X(s; t, x, u(\cdot)), u(s)) ds + h(X(T; t, x, u(\cdot))), \quad (1.3)$$

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for some given maps  $g : [0, T] \times \mathbb{R}^n \times U \rightarrow \mathbb{R}$  and  $h : \mathbb{R}^n \rightarrow \mathbb{R}$ . The terms on the right hand side of (1.3) are referred to as the *running cost* and the *terminal cost*, respectively. The following is a classical optimal control problem.

**Problem (D).** For any given initial pair  $(t, x) \in [0, T] \times \mathbb{R}^n$ , find a  $\bar{u}(\cdot) \in \mathcal{U}[t, T]$  such that

$$J(t, x; \bar{u}(\cdot)) = \inf_{u(\cdot) \in \mathcal{U}[t, T]} J(t, x; u(\cdot)). \quad (1.4)$$

Any  $\bar{u}(\cdot) \in \mathcal{U}[t, T]$  satisfying the above is called an *optimal control* for  $(t, x)$ ,  $\bar{X}(\cdot) \equiv X(\cdot; t, x, \bar{u}(\cdot))$  is called the corresponding *optimal trajectory*, and  $(\bar{X}(\cdot), \bar{u}(\cdot))$  is referred to as an *optimal pair*.

Note that sometimes we might encounter the following seemingly a little more general form of the cost functional

$$\tilde{J}(t, x; u(\cdot)) = \int_t^T e^{-\int_t^s c(r, X(r), u(r))dr} g(s, X(s), u(s))ds + e^{-\int_t^T c(r, X(r), u(r))dr} h(X(T)), \quad (1.5)$$

with  $c(\cdot)$  being some map taking nonnegative values, which may be called a *discount map*. A special case is  $c(\cdot) = \delta > 0$ , a positive constant (which is call a *discount rate*). Due to its form, the term  $e^{-\int_t^s c(r, X(r), u(r))dr}$  is called an *exponential discounting*. If we introduce

$$\begin{cases} \dot{X}^0(s) = c(s, X(s), u(s)), & s \in [t, T], \\ X^0(t) = 0, \end{cases} \quad (1.6)$$

and regard  $X^0(\cdot)$  an additional component of the state, then the state equation is augmented by one dimension and the cost functional becomes

$$\tilde{J}(t, x; u(\cdot)) = \int_t^T e^{-X^0(s)} g(s, X(s), u(s))ds + e^{-X^0(T)} h(X(T)). \quad (1.7)$$

which is of form (1.3). Therefore, an optimal control problem with an exponential discounting can be transformed to an optimal control problem without exponential discounting. In another word, containing an exponential discounting in the cost functional does not make the original problem mathematically more general.

Dynamic programming method is a powerful classical approach to Problem (D). This method suggests us define the *value function* of Problem (D) by the following:

$$\begin{cases} V(t, x) = \inf_{u(\cdot) \in \mathcal{U}[t, T]} J(t, x; u(\cdot)), & (t, x) \in [0, T] \times \mathbb{R}^n, \\ V(T, x) = h(x), & \forall x \in \mathbb{R}^n. \end{cases} \quad (1.8)$$

It is well-known that the following Bellman's *principle of optimality* holds ([25]):

$$V(t, x) = \inf_{u(\cdot) \in \mathcal{U}[t, \tau]} \left\{ \int_t^\tau g(s, X(s), u(s))ds + V(\tau, X(\tau)) \right\}, \quad \forall \tau \in [t, T]. \quad (1.9)$$

Now, if  $\bar{u}(\cdot) \in \mathcal{U}[t, T]$  is an optimal control for the initial pair  $(t, x)$ , then, from the above, for any  $\tau \in (t, T)$ ,

$$\begin{aligned} V(t, x) &= J(t, x; \bar{u}(\cdot)) = \int_t^T g(s, \bar{X}(s), \bar{u}(s)) ds + h(\bar{X}(T)) \\ &= \int_t^\tau g(s, \bar{X}(s), \bar{u}(s)) ds + J(\tau, \bar{X}(\tau); \bar{u}|_{[\tau, T]}(\cdot)) \\ &\geq \int_t^\tau g(s, \bar{X}(s), \bar{u}(s)) ds + V(\tau, \bar{X}(\tau)) \geq V(t, x). \end{aligned} \quad (1.10)$$

Hence, all the equalities in the above have to hold. Consequently,

$$\inf_{u(\cdot) \in \mathcal{U}[\tau, T]} J(\tau, \bar{X}(\tau); u(\cdot)) = V(\tau, \bar{X}(\tau)) = J(\tau, \bar{X}(\tau); \bar{u}|_{[\tau, T]}(\cdot)). \quad (1.11)$$

This means that for any  $0 \leq t < \tau < T$ , the restriction  $\bar{u}|_{[\tau, T]}(\cdot) \in \mathcal{U}[\tau, T]$  of optimal control  $\bar{u}(\cdot) \in \mathcal{U}[t, T]$  for the initial pair  $(t, x)$  on the time interval  $[t, T]$  is optimal for the initial pair  $(\tau, \bar{X}(\tau))$ . Such a phenomenon is referred to as the *time-consistency* of Problem (D). The advantage of the time-consistency is that one needs only to solve Problem (D) for a given initial pair  $(t, x) \in [0, T] \times \mathbb{R}^n$ , and as time goes by, the restriction of the optimal control  $\bar{u}(\cdot)$  for  $(t, x)$  on any later time interval  $[\tau, T]$  will automatically be an optimal control for the corresponding initial pair  $(\tau, \bar{X}(\tau))$ .

However, common sense tells us that the time-consistency issue in real life is actually never so simple. There are two main reasons: First, as time goes by, the environment (in the broad sense) is changing, for example, invention of new technology, new limits of resource allocation, etc., and therefore the controlled system has to be modified according to the new initial pairs; and secondly, people keep changing minds/objectives, which leads to the change of cost functional. Due to these changes, one expects some dramatic changes in the formulation of optimal control problems, as well as the solutions to the problems.

To make our statement more appealing from mathematical point of view, let us look at a very simple illustrative example. Consider a one-dimensional controlled ODE:

$$\begin{cases} \dot{X}(s) = u(s), & s \in [t, T], \\ X(t) = x, \end{cases} \quad (1.12)$$

with cost functional

$$J(t, x; u(\cdot)) = \int_t^T u(s)^2 ds + h(t)X(T; t, x, u(\cdot))^2, \quad (1.13)$$

where  $h : [0, T] \rightarrow [\delta, \infty)$ , for some  $\delta > 0$ , and  $U = \mathbb{R}$ . We pose the following optimal control problem.

**Problem (C).** For given  $(t, x) \in [0, T] \times \mathbb{R}$ , find a  $\bar{u}(\cdot) \in \mathcal{U}[t, T]$  such that

$$J(t, x; \bar{u}(\cdot)) = \inf_{u(\cdot) \in \mathcal{U}[t, T]} J(t, x; u(\cdot)). \quad (1.14)$$

Note that the above problem looks like a simple standard linear quadratic optimal control problem (LQ problem, for short), except that the terminal weight  $h(t)$  depends on the parameter  $t$  (which is the initial time of the problem).

It is clear that for any initial pair  $(t, x) \in [0, T] \times \mathbb{R}$ ,  $u(\cdot) \mapsto J(t, x; u(\cdot))$  is convex and coercive. Thus, there exists a unique optimal control for Problem (C). We can show that (see the Appendix) the optimal control of Problem (C) is given by

$$\bar{u}(s) \equiv \bar{u}(s; t, x) = -\frac{xh(t)}{1 + h(t)(T - t)}, \quad s \in [t, T], \quad (1.15)$$

and the corresponding optimal trajectory is given by

$$\bar{X}(s; t, x, \bar{u}(\cdot)) = x \frac{1 + h(t)(T - s)}{1 + h(t)(T - t)}, \quad s \in [t, T]. \quad (1.16)$$

Now, for  $\tau \in (t, T)$ , we consider Problem (C) on  $[\tau, T]$  with initial state

$$y = \bar{X}(\tau; t, x, \bar{u}(\cdot)) = x \frac{1 + h(t)(T - \tau)}{1 + h(t)(T - t)}. \quad (1.17)$$

We can show that

$$J(\tau, y; \bar{u}(\cdot)) - \inf_{u(\cdot) \in \mathcal{U}[\tau, T]} J(\tau, y; u(\cdot)) = \frac{x^2[h(\tau) - h(t)]^2(T - \tau)}{[1 + h(\tau)(T - t)][1 + h(t)(T - \tau)]^2}. \quad (1.18)$$

This tells us that the restriction of  $\bar{u}(\cdot; t, x)$  on  $[\tau, T]$  is not optimal for Problem (C) with initial pair  $(\tau, \bar{X}(\tau; t, x))$ , in general. Such a phenomenon is called *time-inconsistency*.

Qualitative analysis on time-inconsistent behaviors can at least be traced back to the works by Hume [10] in 1739 and by Smith [22] in 1759. Later relevant works were made by Malthus [14] in 1828, Jevons [11] in 1871, Marshall [16] in 1890, Böhm-Bawerk [4] in 1891, and Pareto [19] in 1909, and so on. Mathematical formulation of time-inconsistency was firstly presented by Strotz [23] in 1955, followed by Pollak [21], Peleg-Yaari [20], Goldman [7], Laibson [13], etc. See Palacios-Huerta [18] for an interesting survey on the history. The above-mentioned mathematical works, starting from Strotz, mainly studied problems for either discrete dynamic systems or simple ODEs, involving non-exponential discounting, meaning that in the cost functional (see (1.5) with  $c(\cdot) = \delta$ ), the classical exponential discounting  $e^{-\delta(s-t)}$  is replaced by a function  $h(s-t)$ . Recently, Ekeland-Lazrak [5] and Ekeland-Pirvu [6] continued the study of non-exponential discounting problems both for simple ODEs and SDEs. At the same time, Basak-Chabakauri [1] and Björk-Murgoci [3] started to discuss the problems with the cost/payoff functional depending on the initial pair  $(t, x)$ . We refer to [8], [9], [12], [17], [24] for some relevant results.

In general, for any given initial pair  $(t, x) \in [0, T] \times \mathbb{R}^n$ , we can consider the following controlled system:

$$\begin{cases} \dot{X}(s) = b(t, x, s, X(s), u(s)), & s \in [t, T], \\ X(t) = x, \end{cases} \quad (1.19)$$

with the cost functional:

$$J(t, x; u(\cdot)) = \int_t^T g(t, x, s, X(s), u(s)) ds + h(t, x, X(T)). \quad (1.20)$$

We point out that state equation (1.19) and cost functional (1.20) are significantly different from (1.1) and (1.3), respectively, due to the way they depend on the initial pair  $(t, x) \in [0, T] \times \mathbb{R}^n$ .

Such a dependence allows us to catch some situations that people will modify the control system and/or the cost functional at different initial pair. Clearly, our setting is much more general than [5]. Naturally, one could pose the following optimal control problem.

**Problem (N).** For  $(t, x) \in [0, T] \times \mathbb{R}^n$ , find  $\bar{u}(\cdot) \in \mathcal{U}[t, T]$  such that

$$J(t, x; \bar{u}(\cdot)) = \inf_{u(\cdot) \in \mathcal{U}[t, T]} J(t, x; u(\cdot)). \quad (1.21)$$

It is clear that Problem (C) is a special case of Problem (N). Hence, Problem (N) is time-inconsistent, in general. Any optimal control  $\bar{u}(\cdot) \in \mathcal{U}[t, T]$  of Problem (N) is referred to as a *pre-committed* optimal control on  $[t, T]$ . Due to the time-inconsistency, finding an optimal control  $\bar{u}(\cdot) \in \mathcal{U}[t, T]$  for Problem (N) (assuming it exists) might not be very useful (if it is not useless) in long run. Hence, Problem (N) is natural, but is a little too naive.

In this paper, we will concentrate on a linear-quadratic time-inconsistent control problem. We will present a time-consistent solution via a “sophisticated” approach. The main idea comes from the works [23], [21], [20], and [7]. Here is a brief description. Take a partition  $\Delta : 0 = t_0 < t_1 < \dots < t_N = T$  of the time interval  $[0, T]$ . Consider an  $N$ -person non-cooperative differential game: the  $k$ -th player (which may be called self- $k$ ) starts the game from the initial pair  $(t_{k-1}, X(t_{k-1}))$  and controls the system on  $[t_{k-1}, t_k]$ , to minimize his own cost functional. At  $t = t_k$ , the next player (the  $(k+1)$ -th player, or self- $(k+1)$ ) takes over, starting from the initial pair  $(t_k, X_k(t_k))$  which is the *terminal pair* of the  $k$ -th player, and controlling the system on  $[t_k, t_{k+1}]$ , etc. Each player knows that the later players will do their best, and will modify their control systems as well as their cost functionals. However, in measuring the performance of the controls, each player will discount the cost/payoff *in his/her own way*. This is the main issue in handling the time-inconsistency, and it also has to be treated this way so that the results can recover those for exponential discounting situations. It is expected that as the mesh size  $\|\Delta\| \equiv \max\{t_k - t_{k-1} \mid 1 \leq k \leq N\} \rightarrow 0$ , the Nash equilibrium strategy to the  $N$ -person differential game should approach to the desired time-consistent solution of the original time-inconsistent Problem (N).

The rest of the paper is organized as follow. In section 2, we collect some preliminary results, mainly some careful estimates relevant to our time-inconsistent optimal control problem. Section 3 is devoted to a study of  $N$ -person differential game. In Section 4, we will discuss the convergence of Nash equilibrium value function for the  $N$ -person differential game, as well as a sufficient condition for the existence of time-consistent equilibrium control for Problem (N). Finally, a time-inconsistent LQ problem will be presented.

## 2 $N$ -Person Differential Games

Consider the following linear controlled ODE parameterized by  $(t, x) \in [0, T] \times \mathbb{R}^n$ :

$$\begin{cases} \dot{X}(s) = A(t, x, s)X(s) + B(t, x, s)u(s), & s \in [t, T], \\ X(t) = x, \end{cases} \quad (2.1)$$

with the cost functional

$$J(t, x; u(\cdot)) = \langle G(t, x)X(T), X(T) \rangle + \int_t^T [\langle Q(t, x, s)X(s), X(s) \rangle + \langle R(t, x, s)u(s), u(s) \rangle] ds. \quad (2.2)$$

Here  $A$ ,  $B$ ,  $Q$ ,  $R$  and  $G$  are some given suitable maps. Let  $\Delta$  be a partition of  $[0, T]$  given by

$$\Delta : 0 = t_0 < t_1 < \cdots < t_N = T.$$

We now introduce an  $N$ -person differential game associated with  $\Delta$ . These  $N$  players are labeled by  $k = 1, 2, \dots, N$ . The  $k$ -th player chooses controls from  $\mathcal{U}[t_{k-1}, t_k]$ . Any  $(u_1(\cdot), \dots, u_N(\cdot)) \in \mathcal{U}[t_0, t_1] \times \cdots \times \mathcal{U}[t_{N-1}, t_N]$  is identified with  $u^\Delta(\cdot) \in \mathcal{U}[0, T]$  where

$$u^\Delta(s) = u_k(s), \quad s \in [t_{k-1}, t_k), \quad 1 \leq k \leq N. \quad (2.3)$$

Now, for any  $(x, u^\Delta(\cdot)) \in \mathbb{R}^n \times \mathcal{U}[0, T]$ , let  $X^\Delta(\cdot)$  be the solution to the following:

$$\begin{cases} \dot{X}^\Delta(s) = A(t_{k-1}, X^\Delta(t_{k-1}), s)X^\Delta(s) + B(t_{k-1}, X^\Delta(t_{k-1}), s)u^\Delta(s), \\ \quad \quad \quad s \in (t_{k-1}, t_k), \quad 1 \leq k \leq N, \\ X^\Delta(0) = x, \end{cases} \quad (2.4)$$

The  $k$ -th player has the following cost functional:

$$\begin{aligned} J_k(u^\Delta(\cdot)) &\equiv J_k(u_1(\cdot), \dots, u_N(\cdot)) = J(t_{k-1}, X^\Delta(t_{k-1}), u^\Delta(\cdot)) \\ &\equiv \langle G(t_{k-1}, X^\Delta(t_{k-1}))X^\Delta(T), X^\Delta(T) \rangle \\ &\quad + \int_{t_{k-1}}^T [\langle Q(t_{k-1}, X^\Delta(t_{k-1}), s)X^\Delta(s), X^\Delta(s) \rangle + \langle R(t_{k-1}, X^\Delta(t_{k-1}), s)u^\Delta(s), u^\Delta(s) \rangle] ds. \end{aligned} \quad (2.5)$$

For any  $x \in \mathbb{R}^n$  and any partition  $\Delta$  of  $[0, T]$ , we now pose the following problem.

**Problem (LQ $^\Delta$ ).** Find a control  $\bar{u}^\Delta(\cdot) \equiv (\bar{u}_1(\cdot), \dots, \bar{u}_N(\cdot)) \in \mathcal{U}[0, T]$  such that for each  $k = 1, 2, \dots, N$ ,

$$\begin{aligned} J_k(\bar{u}^\Delta(\cdot)) &\equiv J_k(\bar{u}_1(\cdot), \dots, \bar{u}_{k-1}(\cdot), \bar{u}_k(\cdot), \bar{u}_{k+1}(\cdot), \dots, \bar{u}_N(\cdot)) \\ &\leq J_k(\bar{u}_1(\cdot), \dots, \bar{u}_{k-1}(\cdot), u_k(\cdot), \bar{u}_{k+1}(\cdot), \dots, \bar{u}_N(\cdot)), \quad \forall u_k(\cdot) \in \mathcal{U}[t_{k-1}, t_k], \end{aligned} \quad (2.6)$$

Any control  $\bar{u}^\Delta(\cdot)$  satisfying the above is called an *equilibrium control* of Problem (LQ $^\Delta$ ). The corresponding state trajectory  $\bar{X}^\Delta(\cdot)$  and the pair  $(\bar{X}^\Delta(\cdot), \bar{u}^\Delta(\cdot))$  are called an *equilibrium state trajectory* and an *equilibrium pair* of Problem (LQ $^\Delta$ ), respectively.

We now introduce the following assumptions.

**(H1)** The maps  $A : [0, T] \times [0, T] \rightarrow \mathbb{R}^{n \times n}$ ,  $B : [0, T] \times [0, T] \rightarrow \mathbb{R}^{n \times m}$ ,  $Q : [0, T] \times [0, T] \rightarrow \mathcal{S}^n$ ,  $R : [0, T] \times [0, T] \rightarrow \mathcal{S}^m$ , and  $G : [0, T] \rightarrow \mathcal{S}^n$  are continuous. There exist constants  $L, \delta > 0$  such that

$$\begin{aligned} &\|A(t, s) - A(r, s)\| + \|B(t, s) - B(r, s)\| + \|Q(t, s) - Q(r, s)\| \\ &+ \|R(t, s) - R(r, s)\| + \|G(t) - G(r)\| \leq L|t - r|, \quad s, t, r \in [0, T]. \end{aligned} \quad (2.7)$$

and

$$Q(t, s), G(t) \geq 0, \quad R(t, s) \geq \delta I, \quad \forall t, s \in [0, T]. \quad (2.8)$$

**(H2)** The maps  $G(\cdot)$ ,  $Q(\cdot, \cdot)$ , and  $R(\cdot, \cdot)$  satisfy the following:

$$G(t) \leq G(r), \quad Q(t, s) \leq Q(r, s), \quad R(t, s) \leq R(r, s), \quad \forall 0 \leq t \leq r \leq s \leq T. \quad (2.9)$$

For any partition  $\Delta$  of  $[0, T]$ , we denote

$$\begin{cases} A^\Delta(s) = \sum_{k=1}^N A(t_{k-1}, s) I_{[t_{k-1}, t_k)}(s), & B^\Delta(s) = \sum_{k=1}^N B(t_{k-1}, s) I_{[t_{k-1}, t_k)}(s), \\ Q^\Delta(s) = \sum_{k=1}^N Q(t_{k-1}, s) I_{[t_{k-1}, t_k)}(s), & R^\Delta(s) = \sum_{k=1}^N R(t_{k-1}, s) I_{[t_{k-1}, t_k)}(s). \end{cases}$$

Our first result is the following.

**Theorem 2.1.** Let (H1) hold. For any partition  $\Delta$  of  $[0, T]$  and any  $x \in \mathbb{R}^n$ , Problem (LQ $^\Delta$ ) admits a unique equilibrium pair  $(\bar{X}^\Delta(\cdot), \bar{u}^\Delta(\cdot))$ . Moreover,  $\bar{X}^\Delta(\cdot)$  and  $\bar{u}^\Delta(\cdot)$  are linked by the following:

$$\bar{u}^\Delta(s) = -R^\Delta(s)^{-1} B^\Delta(s)^T P^\Delta(s) \bar{X}^\Delta(s), \quad s \in [0, T], \quad (2.10)$$

where  $P^\Delta(\cdot)$  is the unique solution to the following Riccati equation:

$$\begin{cases} \dot{P}^\Delta(s) + P^\Delta(s) A^\Delta(s) + A^\Delta(s)^T P^\Delta(s) + Q^\Delta(s) \\ \quad - P^\Delta(s) B^\Delta(s) R^\Delta(s)^{-1} B^\Delta(s)^T P^\Delta(s) = 0, & s \in (t_{k-1}, t_k), \\ P^\Delta(t_k - 0) = \Phi^\Delta(t_N; t_k)^T G(t_{k-1}) \Phi^\Delta(t_N; t_k) \\ \quad + \int_{t_k}^{t_N} \left( \Phi^\Delta(s; t_k)^T Q(t_{k-1}, s) \Phi^\Delta(s; t_k) + \Psi^\Delta(s; t_k)^T R(t_{k-1}, s) \Psi^\Delta(s; t_k) \right) ds, \\ 1 \leq k \leq N, \end{cases} \quad (2.11)$$

with  $\Phi^\Delta(\cdot; t_k)$  ( $0 \leq k \leq N-1$ ) being the solution to the following:

$$\begin{cases} \dot{\Phi}_s^\Delta(s; t_k) = \left[ A^\Delta(s) - B^\Delta(s) R^\Delta(s)^{-1} B^\Delta(s)^T P^\Delta(s) \right] \Phi^\Delta(s; t_k), & s \in (t_k, T], \\ \Phi^\Delta(t_k; t_k) = I, \end{cases} \quad (2.12)$$

and

$$\Psi^\Delta(s; t_k) = -R^\Delta(s)^{-1} B^\Delta(s) P^\Delta(s) \Phi^\Delta(s; t_k), \quad s \in [t_k, T]. \quad (2.13)$$

The equilibrium state trajectory  $\bar{X}^\Delta(\cdot)$  is the solution to the following closed-loop system:

$$\begin{cases} \dot{\bar{X}}^\Delta(s) = \left[ A^\Delta(s) - B^\Delta(s) R^\Delta(s)^{-1} B^\Delta(s)^T P^\Delta(s) \right] \bar{X}^\Delta(s), & s \in [0, T], \\ \bar{X}^\Delta(0) = x, \end{cases} \quad (2.14)$$

and the equilibrium pair  $(\bar{X}^\Delta(\cdot), \bar{u}^\Delta(\cdot))$  can be explicitly represented by the following:

$$\begin{cases} \bar{X}^\Delta(s) = \Phi^\Delta(s; 0)x, \\ \bar{u}^\Delta(s) = \Psi^\Delta(s; 0)x, \end{cases} \quad s \in [0, T]. \quad (2.15)$$

Moreover,

$$0 \leq P^\Delta(t) \leq P_0^\Delta(t), \quad t \in [0, T], \quad (2.16)$$

where  $P_0^\Delta(\cdot)$  is the unique solution to the following Lyapunov equation:

$$\begin{cases} \dot{P}_0^\Delta(s) + P_0^\Delta(s)A^\Delta(s) + A^\Delta(s)^T P_0^\Delta(s) + Q^\Delta(s) = 0, & s \in (t_{k-1}, t_k), \\ P_0^\Delta(t_k - 0) = \Phi^\Delta(t_N; t_k)^T G(t_{k-1}) \Phi^\Delta(t_N; t_k) \\ \quad + \int_{t_k}^{t_N} [\Phi^\Delta(s; t_k)^T Q(t_{k-1}, s) \Phi^\Delta(s; t_k) + \Psi^\Delta(s; t_k)^T R(t_{k-1}, s) \Psi^\Delta(s; t_k)] ds, \\ 1 \leq k \leq N, \end{cases} \quad (2.17)$$

We point out that the solution  $P^\Delta(\cdot)$  of Riccati equation (2.11) and the solution  $P_0^\Delta(\cdot)$  of Lyapunov equation (2.17) have possible jumps at  $t_k$ ,  $k = 1, 2, \dots, N-1$ .

*Proof.* Let  $x \in \mathbb{R}^n$  and  $\Delta : 0 = t_0 < t_1 < \dots < t_N = T$  be given. Let  $(\bar{X}^\Delta(\cdot), \bar{u}^\Delta(\cdot))$  be an equilibrium pair of Problem (LQ $^\Delta$ ). Then the restriction of which on  $[t_{N-1}, t_N]$  is the optimal pair of the LQ problem for Player  $N$  on  $[t_{N-1}, t_N]$ , with the state equation

$$\begin{cases} \dot{X}^\Delta(s) = A^\Delta(s)X^\Delta(s) + B^\Delta(s)u_N(s), & s \in [t_{N-1}, t_N], \\ X^\Delta(t_{N-1}) = \bar{X}^\Delta(t_{N-1}), \end{cases} \quad (2.18)$$

and with the cost functional

$$\begin{aligned} J_N(\bar{u}_1(\cdot), \dots, \bar{u}_{N-1}(\cdot), u_N(\cdot)) &= \langle G_N X^\Delta(t_N), X^\Delta(t_N) \rangle \\ &+ \int_{t_{N-1}}^{t_N} [\langle Q^\Delta(s)X^\Delta(s), X^\Delta(s) \rangle + \langle R^\Delta(s)u_N(s), u_N(s) \rangle] ds, \end{aligned} \quad (2.19)$$

where  $G_N = G(t_{N-1})$ . To study this LQ problem, we consider the following state equation:

$$\begin{cases} \dot{X}_N(s) = A^\Delta(s)X_N(s) + B^\Delta(s)u_N(s), & s \in [t, t_N], \\ X_N(t) = y, \end{cases} \quad (2.20)$$

where  $(t, y) \in [t_{N-1}, t_N] \times \mathbb{R}^n$ , with the cost functional

$$\begin{aligned} J_N(t, y; u_N(\cdot)) &= \langle G_N X_N(t_N), X_N(t_N) \rangle \\ &+ \int_t^{t_N} [\langle Q^\Delta(s)X_N(s), X_N(s) \rangle + \langle R^\Delta(s)u_N(s), u_N(s) \rangle] ds, \end{aligned} \quad (2.21)$$



For such an LQ problem on  $[t, t_N]$ , under (H1), there exists a unique optimal control which must have the following form:

$$\bar{u}_N(s) = -R^\Delta(s)^{-1}B^\Delta(s)^T P^\Delta(s)\bar{X}_N(s), \quad s \in [t, t_N], \quad (2.22)$$

where  $P^\Delta(\cdot)$  is the unique solution to the following Riccati equation:

$$\begin{cases} \dot{P}^\Delta(s) + P^\Delta(s)A^\Delta(s) + A^\Delta(s)^T P^\Delta(s) + Q^\Delta(s) \\ \quad - P^\Delta(s)B^\Delta(s)R^\Delta(s)^{-1}B^\Delta(s)^T P^\Delta(s) = 0, & s \in (t_{N-1}, t_N), \\ P^\Delta(t_N) = G_N, \end{cases} \quad (2.23)$$

and  $\bar{X}_N(\cdot)$  is the solution to the following closed-loop state equation:

$$\begin{cases} \dot{\bar{X}}_N(s) = [A^\Delta(s) - B^\Delta(s)R^\Delta(s)^{-1}B^\Delta(s)^T P^\Delta(s)]\bar{X}_N(s), & s \in [t, t_N], \\ \bar{X}_N(t) = y. \end{cases} \quad (2.24)$$

Let  $\Phi^\Delta(\cdot; t)$  be the solution to the following: (note that  $t \in [t_{N-1}, t_N]$ )

$$\begin{cases} \Phi_s^\Delta(s; t) = [A^\Delta(s) - B^\Delta(s)R^\Delta(s)^{-1}B^\Delta(s)^T P^\Delta(s)]\Phi^\Delta(s; t), & s \in (t, t_N], \\ \Phi^\Delta(t; t) = I, \end{cases} \quad (2.25)$$

and denote

$$\Psi^\Delta(s; t) = -R^\Delta(s)^{-1}B^\Delta(s)P^\Delta(s)\Phi^\Delta(s; t), \quad s \in [t, t_N]. \quad (2.26)$$

Then the optimal pair  $(\bar{X}_N(\cdot), \bar{u}_N(\cdot))$  of LQ problem (on  $[t, t_N]$ ) admits the following representation:

$$\begin{cases} \bar{X}_N(s) = \Phi^\Delta(s; t)y, \\ \bar{u}_N(s) = \Psi^\Delta(s; t)y, \end{cases} \quad s \in [t, t_N]. \quad (2.27)$$

Further,

$$\begin{aligned} \langle P^\Delta(t)y, y \rangle &= J_N(t, y; \bar{u}_N(\cdot)) = \langle G_N \bar{X}_N(t_N), \bar{X}_N(t_N) \rangle \\ &\quad + \int_t^{t_N} [\langle Q(t_{N-1}, s)\bar{X}_N(s), \bar{X}_N(s) \rangle + \langle R(t_{N-1}, s)\bar{u}_N(s), \bar{u}_N(s) \rangle] ds \\ &= \langle [\Phi^\Delta(t_N; t)^T G_N \Phi^\Delta(t_N; t) + \int_t^{t_N} (\Phi^\Delta(s; t)^T Q(t_{N-1}, s)\Phi^\Delta(s; t) \\ &\quad + \Psi^\Delta(s; t)^T R(t_{N-1}, s)\Psi^\Delta(s; t)) ds] y, y \rangle. \end{aligned} \quad (2.28)$$

Since  $y \in \mathbb{R}^n$  can be arbitrarily chosen, we have

$$\begin{aligned} P^\Delta(t) &= \Phi^\Delta(t_N; t)^T G_N \Phi^\Delta(t_N; t) + \int_t^{t_N} [\Phi^\Delta(s; t)^T Q(t_{N-1}, s)\Phi^\Delta(s; t) \\ &\quad + \Psi^\Delta(s; t)^T R(t_{N-1}, s)\Psi^\Delta(s; t)] ds, \quad t \in (t_{N-1}, t_N]. \end{aligned} \quad (2.29)$$

Also, by the optimality of  $\bar{u}_N(\cdot)$ , we have

$$\begin{aligned} \langle P^\Delta(t)y, y \rangle &= J_N(t, y; \bar{u}_N(\cdot)) \leq J_N(t, y; 0) \\ &= \langle G_N X^0(t_N), X^0(t_N) \rangle + \int_t^{t_N} \langle Q(t_{N-1}, s) X^0(s), X^0(s) \rangle ds = \langle P_0^\Delta(t)y, y \rangle, \end{aligned} \quad (2.30)$$

where  $X^0(\cdot)$  is the solution to the following:

$$\begin{cases} \dot{X}^0(s) = A^\Delta(s)X^0(s), & s \in [t, t_N], \\ X^0(t) = y, \end{cases} \quad (2.31)$$

and  $P_0^\Delta(\cdot)$  is the solution to the following Lyapunov equation:

$$\begin{cases} \dot{P}_0^\Delta(s) + P_0^\Delta(s)A^\Delta(s) + A^\Delta(s)^T P_0^\Delta(s) + Q^\Delta(s) = 0, & s \in (t_{N-1}, t_N), \\ P_0^\Delta(t_N - 0) = G_N, \end{cases} \quad (2.32)$$

which can be represented by the following:

$$P_0^\Delta(t) = \Phi_0^\Delta(t_N; t)^T G_N \Phi_0^\Delta(t_N; t) + \int_t^{t_N} \Phi_0^\Delta(s, t)^T Q(t_{N-1}, s) \Phi_0^\Delta(s; t) ds, \quad t \in [t, t_N], \quad (2.33)$$

with  $\Phi_0^\Delta(\cdot; t)$  being the solution to the following:

$$\begin{cases} \frac{\partial}{\partial s} \Phi_0^\Delta(s; t) = A^\Delta(s) \Phi_0^\Delta(s; t), & s \in [t, t_N], \\ \Phi_0^\Delta(t; t) = I. \end{cases} \quad (2.34)$$

Note that  $\Phi_0^\Delta(\cdot; t)$  can be defined for any  $t \in [0, t_N)$ , which will be used below. Hence,

$$0 \leq P^\Delta(t) \leq P_0^\Delta(t), \quad t \in (t_{N-1}, t_N]. \quad (2.35)$$

It is also clear that the restriction of the equilibrium pair  $(\bar{X}^\Delta(\cdot), \bar{u}^\Delta(\cdot))$  on  $(t_{N-1}, t_N]$  admits the following representation:

$$\begin{cases} \bar{X}^\Delta(s) = \Phi^\Delta(s; t_{N-1}) \bar{X}^\Delta(t_{N-1}), \\ \bar{u}^\Delta(s) = \Psi^\Delta(s; t_{N-1}) \bar{X}^\Delta(t_{N-1}), \end{cases} \quad s \in [t_{N-1}, t_N]. \quad (2.36)$$

Next, for Player  $(N-1)$ , inspired by the above, we consider the following state equation:

$$\begin{cases} \dot{X}_{N-1}(s) = A^\Delta(s)X_{N-1}(s) + B^\Delta(s)u_{N-1}(s), & s \in [t, t_{N-1}], \\ X_{N-1}(t) = y, \end{cases} \quad (2.37)$$

where  $(t, y) \in [t_{N-2}, t_{N-1}) \times \mathbb{R}^n$ . Let us denote

$$\begin{cases} \tilde{X}_{N-1}^\Delta(s) = \Phi^\Delta(s; t_{N-1})X_{N-1}(t_{N-1}), \\ \tilde{u}_{N-1}^\Delta(s) = \Psi^\Delta(s; t_{N-1})X_{N-1}(t_{N-1}), \end{cases} \quad s \in [t_{N-1}, t_N]. \quad (2.38)$$

Thus,  $(\tilde{X}_{N-1}^\Delta(\cdot), \tilde{u}_{N-1}^\Delta(\cdot))$  is the optimal pair for Player  $N$  starting from the initial pair  $(t_{N-1}, X_{N-1}(t_{N-1}))$ . The cost functional for the LQ problem of Player  $(N-1)$  on  $[t, t_{N-1}]$  is taken to be

$$\begin{aligned}
& J_{N-1}(t, y; u_{N-1}(\cdot)) \\
&= \int_t^{t_{N-1}} [\langle Q(t_{N-2}, s)X_{N-1}(s), X_{N-1}(s) \rangle + \langle R(t_{N-2}, s)u_{N-1}(s), u_{N-1}(s) \rangle] ds \\
&\quad + \int_{t_{N-1}}^{t_N} [\langle Q(t_{N-2}, s)\tilde{X}_{N-1}^\Delta(s), \tilde{X}_{N-1}^\Delta(s) \rangle + \langle R(t_{N-2}, s)\tilde{u}_{N-1}^\Delta(s), \tilde{u}_{N-1}^\Delta(s) \rangle] ds \\
&\quad + \langle G(t_{N-2})\tilde{X}^\Delta(t_N), \tilde{X}^\Delta(t_N) \rangle \\
&\equiv \int_t^{t_{N-1}} [\langle Q^\Delta(s)X_{N-1}(s), X_{N-1}(s) \rangle + \langle R^\Delta(s)u_{N-1}(s), u_{N-1}(s) \rangle] ds \\
&\quad + \langle G_{N-1}X_{N-1}(t_{N-1}), X_{N-1}(t_{N-1}) \rangle,
\end{aligned} \tag{2.39}$$

where

$$\begin{aligned}
G_{N-1} &= \Phi^\Delta(t_N; t_{N-1})^T G(t_{N-2}) \Phi^\Delta(t_N; t_{N-1}) \\
&\quad + \int_{t_{N-1}}^{t_N} [\Phi^\Delta(s; t_{N-1})^T Q(t_{N-2}, s) \Phi^\Delta(s; t_{N-1}) + \Psi^\Delta(s; t_{N-1})^T R(t_{N-2}, s) \Psi^\Delta(s; t_{N-1})] ds.
\end{aligned} \tag{2.40}$$

For such an LQ problem (on  $[t, t_{N-1}]$ ), under (H1), the optimal control is given by

$$\bar{u}_{N-1}(s) = -R^\Delta(s)^{-1} B^\Delta(s)^T P^\Delta(s) \bar{X}_{N-1}(s), \quad s \in [t, t_{N-1}], \tag{2.41}$$

where  $P^\Delta(\cdot)$  is the solution to the following Riccati equation:

$$\begin{cases} \dot{P}^\Delta(s) + P^\Delta(s)A^\Delta(s) + A^\Delta(s)^T P^\Delta(s) + Q^\Delta(s) \\ \quad - P^\Delta(s)B^\Delta(s)R^\Delta(s)^{-1}B^\Delta(s)^T P^\Delta(s) = 0, & s \in (t_{N-2}, t_{N-1}), \\ P^\Delta(t_{N-1} - 0) = G_{N-1}, \end{cases} \tag{2.42}$$

and  $\bar{X}_{N-1}(\cdot)$  is the solution to the following closed-loop state equation:

$$\begin{cases} \dot{\bar{X}}_{N-1}(s) = [A^\Delta(s) - B^\Delta(s)R^\Delta(s)^{-1}B^\Delta(s)^T P^\Delta(s)] \bar{X}_{N-1}(s), & s \in [t, t_{N-1}], \\ \bar{X}_{N-1}(t) = y. \end{cases} \tag{2.43}$$

Now, similar to (2.25), for  $t \in [t_{N-2}, t_{N-1}]$ , let  $\Phi^\Delta(\cdot; t)$  be the solution to the following:

$$\begin{cases} \Phi_s^\Delta(s; t) = [A^\Delta(s) - B^\Delta(s)R^\Delta(s)^{-1}B^\Delta(s)^T P^\Delta(s)] \Phi^\Delta(s; t), & s \in (t, t_N], \\ \Phi^\Delta(t; t) = I, \end{cases} \tag{2.44}$$

and denote

$$\Psi^\Delta(s; t) = -R^\Delta(s)^{-1} B^\Delta(s) P^\Delta(s) \Phi^\Delta(s; t), \quad s \in [t, t_N]. \tag{2.45}$$

It is clear that for  $t \in [t_{N-2}, t_{N-1}]$ ,

$$\begin{cases} \Phi^\Delta(s; t) = \Phi^\Delta(s; t_{N-1})\Phi^\Delta(t_{N-1}; t), \\ \Psi^\Delta(s; t) = \Psi^\Delta(s; t_{N-1})\Phi^\Delta(t_{N-1}; t), \end{cases} \quad s \in [t_{N-1}, t_N], \quad (2.46)$$

and the optimal pair  $(\bar{X}_{N-1}(\cdot), \bar{u}_{N-1}(\cdot))$  of the LQ problem associated with (2.37) and (2.39) (on  $[t, t_{N-1}]$ ) is given by the following:

$$\begin{cases} \bar{X}_{N-1}(s) = \Phi^\Delta(s; t)y, \\ \bar{u}_{N-1}(s) = \Psi^\Delta(s; t)y, \end{cases} \quad s \in [t, t_{N-1}]. \quad (2.47)$$

Hence, the restriction of the equilibrium pair  $(\bar{X}^\Delta(\cdot), \bar{u}^\Delta(\cdot))$  on  $[t_{N-2}, t_N]$  admits the following representation:

$$\begin{cases} \bar{X}^\Delta(s) = \Phi^\Delta(s; t_{N-2})\bar{X}^\Delta(t_{N-2}), \\ \bar{u}^\Delta(s) = \Psi^\Delta(s; t_{N-2})\bar{X}^\Delta(t_{N-2}), \end{cases} \quad s \in [t_{N-2}, t_N]. \quad (2.48)$$

Further,

$$\begin{aligned} \langle P^\Delta(t)y, y \rangle &= J_{N-1}(t, y; \bar{u}_{N-1}(\cdot)) = \langle G_{N-1}\bar{X}_{N-1}(t_{N-1}), \bar{X}_{N-1}(t_{N-1}) \rangle \\ &\quad + \int_t^{t_{N-1}} [\langle Q(t_{N-2}, s)\bar{X}_{N-1}(s), \bar{X}_{N-1}(s) \rangle + \langle R(t_{N-2}, s)\bar{u}_{N-1}(s), \bar{u}_{N-1}(s) \rangle] ds \\ &= \langle [\Phi^\Delta(t_{N-1}; t)^T G_{N-1} \Phi^\Delta(t_{N-1}; t) + \int_t^{t_{N-1}} [\Phi^\Delta(s; t)^T Q(t_{N-2}, s) \Phi^\Delta(s; t) \\ &\quad + \Psi^\Delta(s; t)^T R(t_{N-2}, s) \Psi^\Delta(s; t)] ds] y, y \rangle \\ &= \langle [\Phi^\Delta(t_N; t)^T G(t_{N-2}) \Phi^\Delta(t_N; t) + \int_t^{t_N} (\Phi^\Delta(s; t)^T Q(t_{N-2}, s) \Phi^\Delta(s; t) \\ &\quad + \Psi^\Delta(s; t)^T R(t_{N-2}, s) \Psi^\Delta(s; t)) ds] y, y \rangle. \end{aligned} \quad (2.49)$$

Since  $y \in \mathbb{R}^n$  can be arbitrarily chosen, we have

$$\begin{aligned} P^\Delta(t) &= \Phi^\Delta(t_N; t)^T G(t_{N-2}) \Phi^\Delta(t_N; t) + \int_t^{t_N} [\Phi^\Delta(s; t)^T Q(t_{N-2}, s) \Phi^\Delta(s; t) \\ &\quad + \Psi^\Delta(s; t)^T R(t_{N-2}, s) \Psi^\Delta(s; t)] ds, \quad t \in (t_{N-2}, t_{N-1}). \end{aligned} \quad (2.50)$$

Also, by the optimality of  $\bar{u}_{N-1}(\cdot)$ , we have

$$\begin{aligned} \langle P^\Delta(t)y, y \rangle &= J_{N-1}(t, y; \bar{u}_{N-1}(\cdot)) \leq J_{N-1}(t, y; 0) \\ &= \langle G_{N-1}X^0(t_{N-1}), X^0(t_{N-1}) \rangle + \int_t^{t_{N-1}} \langle Q(t_{N-2}, s)X^0(s), X^0(s) \rangle ds = \langle P_0^\Delta(t)y, y \rangle, \end{aligned} \quad (2.51)$$

where  $X^0(\cdot)$  is the solution to the following:

$$\begin{cases} \dot{X}^0(s) = A^\Delta(s)X^0(s), & s \in [t, t_{N-1}], \\ X^0(t) = y, \end{cases} \quad (2.52)$$

and  $P_0^\Delta(\cdot)$  is the solution to the following Lyapunov equation:

$$\begin{cases} \dot{P}_0^\Delta(s) + P_0^\Delta(s)A^\Delta(s) + A^\Delta(s)^T P_0^\Delta(s) + Q^\Delta(s) = 0, & s \in (t_{N-2}, t_{N-1}), \\ P_0^\Delta(t_{N-1} - 0) = G_{N-1}, \end{cases} \quad (2.53)$$

which, similar to the above, admits the following representation:

$$P_0^\Delta(t) = \Phi_0^\Delta(t_{N-1}; t)^T G_{N-1} \Phi_0^\Delta(t_{N-1}; t) + \int_t^{t_{N-1}} \Phi_0^\Delta(s, t)^T Q^\Delta(s) \Phi_0^\Delta(s; t) ds, \quad t \in [t, t_{N-1}], \quad (2.54)$$

Hence,

$$0 \leq P^\Delta(t) \leq P_0^\Delta(t), \quad t \in (t_{N-2}, t_{N-1}). \quad (2.55)$$

Then one can apply induction to complete the proof.  $\square$

### 3 Time-Consistent Solutions

We now pose the following problem.

**Problem (LQ).** For any given  $x \in \mathbb{R}^n$ , find a control  $\bar{u}(\cdot) \in \mathcal{U}[0, T]$  satisfying the following: For any  $\varepsilon > 0$ , there exists a  $\delta > 0$  such that for any partition  $\Delta$  of  $[0, T]$  with  $\|\Delta\| < \delta$ , one has

$$J_k(\bar{u}(\cdot)) \leq J_k(\bar{u}^\Delta(\cdot)) + \varepsilon. \quad (3.1)$$

Any control  $\bar{u}(\cdot) \in \mathcal{U}[0, T]$  satisfying the above is called an *equilibrium control* of Problem (LQ). The corresponding state trajectory  $\bar{X}(\cdot)$  and the pair  $(\bar{X}(\cdot), \bar{u}(\cdot))$  are called an *equilibrium state trajectory* and an *equilibrium pair* of Problem (LQ), respectively.

The following gives a weaker notion of time-consistent solutions to Problem (LQ).

**Definition 3.1.** A control  $\bar{u}(\cdot) \in \mathcal{U}[0, T]$  is called a *weak equilibrium control* of Problem (LQ) if there exists a sequence of partitions  $\Delta_m$  of  $[0, T]$  with  $\|\Delta_m\| \rightarrow 0$  so that for any  $\varepsilon > 0$ , there exists an  $m_0 > 0$  such that

$$J_k(\bar{u}(\cdot)) \leq J_k(\bar{u}^{\Delta_m}(\cdot)) + \varepsilon, \quad \forall m \geq m_0. \quad (3.2)$$

Our next goal is to find the limit as the mesh size  $\|\Delta\|$  of  $\Delta$  approaches to zero. For this, we need (H2).

**Theorem 3.1.** Let (H1)–(H2) hold. Then for any partition  $\Delta$  of  $[0, T]$ ,

$$0 \leq P^\Delta(t) \leq P_0^\Delta(t) \leq \bar{P}_0^\Delta(t), \quad t \in [0, T], \quad (3.3)$$

where  $\bar{P}_0^\Delta(\cdot)$  is the unique solution to the following Lyapunov equation:

$$\begin{cases} \dot{\bar{P}}_0^\Delta(s) + \bar{P}_0^\Delta(s)A^\Delta(s) + A^\Delta(s)^T \bar{P}_0^\Delta(s) + Q^\Delta(s) = 0, & s \in (0, t_N), \\ \bar{P}_0^\Delta(t_N) = G(t_{N-1}). \end{cases} \quad (3.4)$$

Consequently,  $P^\Delta(\cdot)$  is bounded uniformly in  $\Delta$ .

*Proof.* Recall that for  $k = 1, 2, \dots, N-1$ ,

$$\begin{aligned} P_0^\Delta(t_k - 0) &= \Phi^\Delta(t_N; t_k)^T G(t_{k-1}) \Phi^\Delta(t_N; t_k) \\ &\quad + \int_{t_k}^{t_N} [\Phi^\Delta(s; t_k)^T Q(t_{k-1}, s) \Phi^\Delta(s; t_k) + \Psi^\Delta(s; t_k)^T R(t_{k-1}, s) \Psi^\Delta(s; t_k)] ds, \end{aligned} \quad (3.5)$$

and (making use of the monotonicity of  $t \mapsto G(t)$ ,  $t \mapsto Q(s, t)$ , and  $t \mapsto R(s, t)$ )

$$\begin{aligned} P_0^\Delta(t_k + 0) &\geq P^\Delta(t_k + 0) = \Phi^\Delta(t_N; t_k)^T G(t_k) \Phi^\Delta(t_N; t_k) \\ &\quad + \int_{t_k}^{t_N} [\Phi^\Delta(s; t_k)^T Q(t_k, s) \Phi^\Delta(s; t_k) + \Psi^\Delta(s; t_k)^T R(t_k, s) \Psi^\Delta(s; t_k)] ds \\ &\geq \Phi^\Delta(t_N; t_k)^T G(t_{k-1}) \Phi^\Delta(t_N; t_k) + \int_{t_k}^{t_N} [\Phi^\Delta(s; t_k)^T Q(t_{k-1}, s) \Phi^\Delta(s; t_k) \\ &\quad + \Psi^\Delta(s; t_k)^T R(t_{k-1}, s) \Psi^\Delta(s; t_k)] ds = P^\Delta(t_k - 0) = P_0^\Delta(t_k - 0). \end{aligned} \quad (3.6)$$

Note that

$$P_0^\Delta(t) = \bar{P}_0^\Delta(t), \quad t \in (t_{N-1}, t_N], \quad (3.7)$$

and due to (making use of (3.6) for  $k = N-1$ )

$$P_0^\Delta(t_{N-1} - 0) \leq P^\Delta(t_{N-1} + 0) \leq P_0^\Delta(t_{N-1} + 0) = \bar{P}_0^\Delta(t_{N-1}) = \bar{P}_0^\Delta(t_{N-1} - 0), \quad (3.8)$$

we have

$$P_0^\Delta(t) \leq \bar{P}_0^\Delta(t), \quad t \in (t_{N-2}, t_{N-1}). \quad (3.9)$$

Then by induction, we can obtain

$$P_0^\Delta(t) \leq \bar{P}_0^\Delta(t), \quad t \in [0, t_N]. \quad (3.10)$$

By the boundness of  $A(\cdot, \cdot)$  and  $Q(\cdot, \cdot)$ , we have the boundness of  $\bar{P}_0^\Delta(\cdot)$  uniformly in  $\Delta$ . Hence, we complete the proof.  $\square$

We see that  $P^\Delta(\cdot)$  has a possible jump at each  $t_k$ , with the jump size:

$$\begin{aligned} \Delta P^\Delta(t_k) &\equiv P^\Delta(t_k + 0) - P^\Delta(t_k - 0) \\ &= \Phi^\Delta(t_N; t_k)^T [G(t_k) - G(t_{k-1})] \Phi^\Delta(t_N; t_k) \\ &\quad + \int_{t_k}^{t_N} [\Phi^\Delta(s; t_k)^T (Q(t_k, s) - Q(t_{k-1}, s)) \Phi^\Delta(s; t_k) \\ &\quad + \Psi^\Delta(s; t_k)^T (R(t_k, s) - R(t_{k-1}, s)) \Psi^\Delta(s; t_k)] ds \geq 0. \end{aligned} \quad (3.11)$$

By (H1)–(H2), we have

$$\|\Delta P^\Delta(t_k)\| \leq K(t_k - t_{k-1}) \leq K\|\Delta\|. \quad (3.12)$$

Next, we define  $\tilde{P}^\Delta(\cdot)$  as follows:

$$\begin{cases} \tilde{P}^\Delta(t) = P^\Delta(t), & t \in (t_{N-1}, t_N], \\ \tilde{P}^\Delta(t) = P^\Delta(t) + \frac{t - t_{k-1}}{t_k - t_{k-1}} \Delta P^\Delta(t_k), & t \in (t_{k-1}, t_k), \\ \tilde{P}^\Delta(t_k) = P^\Delta(t_k + 0), & 1 \leq k \leq N-1. \end{cases} \quad (3.13)$$

Then  $\{\tilde{P}^\Delta(\cdot)\}$  is uniformly bounded and equicontinuous. Hence, we may assume that along a certain sequence  $\Delta_m$  with  $\|\Delta_m\| \rightarrow 0$ ,

$$\lim_{m \rightarrow \infty} \tilde{P}^{\Delta_m}(\cdot) = P(\cdot), \quad (3.14)$$

for some  $P(\cdot)$ . Also, we have

$$\|\tilde{P}^\Delta(t) - P^\Delta(t)\| \leq \max_{1 \leq k \leq N-1} \|\Delta P^\Delta(t_k)\| \leq K \|\Delta\| \rightarrow 0, \quad \text{as } \|\Delta\| \rightarrow 0. \quad (3.15)$$

Hence, we have

$$\lim_{m \rightarrow \infty} \|P^{\Delta_m}(\cdot) - P(\cdot)\| = 0. \quad (3.16)$$

Next, it is clear that

$$\lim_{\|\Delta\| \rightarrow 0} \left\{ \|A^\Delta(s) - A(s, s)\| + \|B^\Delta(s) - B(s, s)\| + \|Q^\Delta(s) - Q(s, s)\| + \|R^\Delta(s) - R(s, s)\| \right\} = 0. \quad (3.17)$$

Hence,

$$\lim_{\|\Delta\| \rightarrow 0} \|\Phi^\Delta(s; t) - \Phi(s; t)\| = 0, \quad (3.18)$$

with  $\Phi(\cdot; t)$  being the solution to the following:

$$\begin{cases} \Phi_s(s; t) = \left[ A(s, s) - B(s, s)R(s, s)^{-1}B(s, s)^T P(s) \right] \Phi(s; t), & s \in (t, t_N], \\ \Phi(t; t) = I. \end{cases} \quad (3.19)$$

Consequently,  $P(\cdot)$  satisfies the following:

$$\begin{aligned} P(t) = & \Phi(T; t)^T G(t) \Phi(T; t) + \int_t^T [\Phi(s; t)^T Q(t, s) \Phi(s; t) \\ & + \Phi(s; t)^T P(s) B(s, s)^T R(s, s)^{-1} R(t, s) R(s, s)^{-1} B(s, s) P(s) \Phi(s; t)] ds, \quad t \in (0, T). \end{aligned} \quad (3.20)$$

Denote

$$A(s) = A(s, s), \quad B(s) = B(s, s), \quad R(s) = R(s, s).$$

Then we have the following system of forward-backward Volterra integral equations:

$$\begin{cases} \Phi(s; t) = I + \int_t^s [A(r) - B(r)R(r)^{-1}B(r)^T P(r)] \Phi(r; t) dr, & s \in [t, T], \\ P(t) = \Phi(T; t)^T G(t) \Phi(T; t) + \int_t^T [\Phi(r; t)^T Q(t, r) \Phi(r; t) \\ \quad + \Phi(r; t)^T P(r) B(r)^T R(r)^{-1} R(t, r) R(r)^{-1} B(r) P(r) \Phi(r; t)] dr, & t \in [0, T]. \end{cases} \quad (3.21)$$

Suppose the above admits a unique solution  $(\Phi(\cdot; \cdot), P(\cdot))$ . Then

$$\lim_{\|\Delta\| \rightarrow 0} \|P^\Delta(\cdot) - P(\cdot)\| = 0, \quad (3.22)$$

and

$$\lim_{\|\Delta\| \rightarrow 0} \|\Phi^\Delta(\cdot; \cdot) - \Phi(\cdot; \cdot)\| = 0. \quad (3.23)$$

Then

$$\lim_{\|\Delta\| \rightarrow 0} \left\{ \|\bar{X}^\Delta(\cdot) - \bar{X}(\cdot)\| + \|\bar{u}^\Delta(\cdot) - \bar{u}(\cdot)\| \right\} = 0, \quad (3.24)$$

with

$$\begin{cases} \bar{X}(s) = \Phi(s; 0)x, \\ \bar{u}(s) = -R(s)^{-1}B(s)P(s)\bar{X}(s) \equiv -R(s)^{-1}B(s)P(s)\Phi(s; 0)x, \end{cases} \quad s \in [0, T]. \quad (3.25)$$

Hence, for any  $\varepsilon > 0$ , there exists a  $\delta > 0$  such that for any partition  $\Delta$  of  $[0, T]$ , as long as  $\|\Delta\| < \delta$ , one has

$$J_k(\bar{u}(\cdot)) \leq J_k(\bar{u}^\Delta(\cdot)) + K\|\Delta\| < J_k(\bar{u}^\Delta(\cdot)) + \varepsilon. \quad (3.26)$$

## References

- [1] S. Basak and G. Chabakauri, *Dynamic mean-variance asset allocation*, Preprint, London Business School, 2008.
- [2] L. D. Berkovitz, *Optimal Control Theory*, Springer-Verlag, New York, 1974.
- [3] T. Björk and A. Murgoci, *A general theory of Markovian time inconsistent stochastic control problem*, working paper.
- [4] E. V. Böhm-Bawerk, *The positive Theory of Capital*, Books for Libraries Press, Freeport, New York 1891.
- [5] I. Ekeland and A. Lazrak, *Being serious about non-commitment: subgame perfect equilibrium in continuous time*, preprint, Univ. British Columbia, 2008.
- [6] I. Ekeland and T. Privu, *Investment and consumption without commitment*, preprint, Univ. British Columbia, 2007.
- [7] S. M. Goldman, *Consistent plans*, *Review of Economic Studies*, 47 (1980), 533–537.
- [8] S. R. Grenadier and N. Wang, *Investment under uncertainty and time-inconsistent preferences*, preprint.
- [9] P. J. Herings and K. I. M. Rohde, *Time-inconsistent preferences in a general equilibrium model*, preprint.
- [10] D. Hume, *A Treatise of Human Nature*, First Edition, 1739; Reprint, Oxford Univ. Press, New York, 1978.



- [11] W. S. Jevons, *Theory of Political Economy*, Mcmillan, London, 1871.
- [12] P. Krusell and A. A. Smith, Jr., *Consumption and saving decisions with quasi-geometric discounting*, *Econometrica*, 71 (2003), 366–375.
- [13] D. Laibson, *Golden eggs and hyperbolic discounting*, *Quarterly J. Econ.*, 112 (1997), 443–477.
- [14] A. Malthus, *An essay on the principle of population, 1826; The Works of Thomas Robert Malthus*, Vols. 2–3, Edited by E. A. Wrigley and D. Souden, W. Pickering, London, 1986.
- [15] J. Marin-Solano and J. Navas, *Non-constant discounting in finite horizon: the free terminal time case*, *J. Economic Dynamics and Control*, 33 (2009), 666–675.
- [16] A. Marshall. *Principles of Economics*, 1st ed., 1890; 8th ed., Macmillan, London, 1920.
- [17] M. Miller and M. Salmon, *Dynamic games and the time inconsistency of optimal policy in open economics*, *The Economic Journal*, 95 (1985), 124–137.
- [18] I. Palacios-Huerta, *Time-inconsistent preferences in Adam Smith and Davis Hume*, *History of Political Economy*, 35 (2003), 241–268.
- [19] V. Pareto, *Manuel d’économie politique*, Girard and Brieve, Paris, 1909.
- [20] B. Peleg and M. E. Yaari, *On the existence of a consistent course of action when tastes are changing*, *Review of Economic Studies*, 40 (1973), 391–401.
- [21] R. A. Pollak, *Consistent planning*, *Review of Economic Studies*, 35 (1968), 185–199.
- [22] A. Smith, *The Theory of Moral Sentiments*, First Edition, 1759; Reprint, Oxford Univ. Press, 1976.
- [23] R. H. Strotz, *Myopia and inconsistency in dynamic utility maximization*, *Review of Econ. Studies*, 23 (1955), 165–180.
- [24] L. Tesfatsion, *Time inconsistency of benevolent government economics*, *J. Public Economics*, 31 (1986), 25–52.
- [25] J. Yong, and X. Y. Zhou, *Stochastic Controls: Hamiltonian Systems and HJB Equations*, Springer-Verlag, New York, 1999.

## Appendix.

Let us now solve Problem (C) explicitly. For any given  $(t, x) \in [0, T] \times \mathbb{R}^n$ , according to a standard LQ theory, in the current case, the corresponding Riccati differential equation reads

$$\begin{cases} \dot{P}(s) - P(s)^2 = 0, & s \in [t, T], \\ P(T) = h(t), \end{cases} \quad (A.1)$$

Clearly the solution of the above Riccati equation depends on  $t$ . Hence, we denote it by  $P(\cdot; t)$ . Simple calculation shows that

$$P(s; t) = \frac{h(t)}{1 + h(t)(T - s)}, \quad s \in [t, T]. \quad (A.2)$$

The optimal control trajectory is the solution to the following closed-loop system

$$\begin{cases} \dot{\bar{X}}(s) = -P(s; t)\bar{X}(s), & s \in [t, T], \\ \bar{X}(t) = x, \end{cases} \quad (A.3)$$

which is given by

$$\bar{X}(s; t, x) = x \frac{1 + h(t)(T - s)}{1 + h(t)(T - t)}, \quad s \in [t, T], \quad (A.4)$$

and the optimal control is given by

$$\bar{u}(s; t, x) = -P(s; t)\bar{X}(s; t, x) = -\frac{xh(t)}{1 + h(t)(T - t)}, \quad s \in [t, T]. \quad (A.5)$$

Now, if we let

$$J(t; \tau, y; u(\cdot)) = \int_{\tau}^T u(s)^2 ds + h(t)X(T; \tau, y, u(\cdot))^2, \quad \tau \in [t, T], \quad (A.6)$$

then the optimal value function (for fixed  $t$ ) is given by

$$\begin{aligned} V(t; \tau, y) &\equiv \inf_{u(\cdot) \in \mathcal{U}[\tau, T]} J(t; \tau, y; u(\cdot)) = J(t; \tau, y; \bar{u}(\cdot)) = P(\tau; t)y^2 \\ &= \frac{h(t)}{1 + h(t)(T - \tau)}y^2, \quad \forall (\tau, y) \in [t, T] \times \mathbb{R}. \end{aligned} \quad (A.7)$$

Next, let  $\tau \in (t, T)$ , we consider Problem (C) on  $[\tau, T]$  with initial state

$$y = \bar{X}(\tau; t, x) = x \frac{1 + h(t)(T - \tau)}{1 + h(t)(T - t)}. \quad (A.8)$$

The same as above, we see that the corresponding solution to the Riccati equation is given by

$$P(s; \tau) = \frac{h(\tau)}{1 + h(\tau)(T - s)}, \quad s \in [\tau, T], \quad (A.9)$$

and

$$\begin{aligned} \inf_{u(\cdot) \in \mathcal{U}[\tau, T]} J(\tau; \tau, y; u(\cdot)) &= P(\tau; \tau)y^2 = \frac{h(\tau)y^2}{1 + h(\tau)(T - \tau)} \\ &= \frac{x^2 h(\tau) [1 + h(t)(T - \tau)]^2}{[1 + h(\tau)(T - \tau)][1 + h(t)(T - t)]^2}. \end{aligned} \quad (A.10)$$

However,

$$\begin{aligned} J(\tau, y; \bar{u}(\cdot)) &= \int_{\tau}^T \bar{u}(s)^2 ds + h(\tau)X(T; \tau, y, \bar{u}(\cdot))^2 \\ &= \frac{x^2 h(t)^2 (T - \tau)}{[1 + h(t)(T - t)]^2} + h(\tau) \left[ y - \frac{xh(t)(T - \tau)}{1 + h(t)(T - t)} \right]^2 \\ &= \frac{x^2 h(t)^2 (T - \tau)}{[1 + h(t)(T - t)]^2} + \frac{x^2 h(\tau)}{[1 + h(t)(T - t)]^2}. \end{aligned} \quad (A.11)$$

Hence,

$$\begin{aligned}
& J(\tau, y; \bar{u}(\cdot)) - \inf_{u(\cdot) \in \mathcal{U}[\tau, T]} J(\tau, y; u(\cdot)) \\
&= \frac{x^2 h(t)^2 (T - \tau) + x^2 h(\tau)}{[1 + h(t)(T - t)]^2} - \frac{x^2 h(\tau) [1 + h(t)(T - \tau)]^2}{[1 + h(\tau)(T - \tau)][1 + h(t)(T - t)]^2} \\
&= x^2 \left[ \frac{[h(t)^2 (T - \tau) + h(\tau)][1 + h(\tau)(T - \tau)] - h(\tau)[1 + h(t)(T - \tau)]^2}{[1 + h(\tau)(T - \tau)][1 + h(t)(T - t)]^2} \right] \\
&= \frac{x^2 [h(\tau) - h(t)]^2 (T - \tau)}{[1 + h(\tau)(T - \tau)][1 + h(t)(T - t)]^2} > 0, \quad \text{unless } h(\tau) = h(t) \text{ or } x = 0.
\end{aligned} \tag{A.12}$$

This shows that the restriction of  $\bar{u}(\cdot; t, x)$  on  $[\tau, T]$  is not necessarily optimal for Problem (C) with initial pair  $(\tau, \bar{X}(\tau; t, x))$ . Hence, Problem (C) is time-inconsistent.

Consider

$$\begin{cases} \dot{X}(s) = u(s), & s \in [t, T], \\ X(t) = x, \end{cases} \quad (3.27)$$

with cost functional

$$J(t, x; u(\cdot)) = \int_t^T u(s)^2 ds + h(t)X(T; t, x, u(\cdot))^2, \quad (3.28)$$

Let  $\Delta : 0 = t_0 < t_1 < t_2 < \dots < t_{N-1} < t_N = T$  be a partition of  $[0, T]$ . Consider an LQ problem on  $[t_{N-1}, t_N]$ , with the state equation

$$\begin{cases} \dot{X}_N(s) = u_N(s), & s \in [t_{N-1}, t_N], \\ X_N(t_{N-1}) = x, \end{cases} \quad (3.29)$$

and cost functional

$$J_N(t_{N-1}, x; u_N(\cdot)) = \int_{t_{N-1}}^{t_N} u_N(s)^2 ds + h(t_{N-1})X_N(t_N)^2. \quad (3.30)$$

The corresponding Riccati differential equation reads

$$\begin{cases} \dot{P}^N(s) - P^N(s)^2 = 0, & s \in [t_{N-1}, t_N], \\ P^N(t_N) = h(t_{N-1}), \end{cases} \quad (A.1)$$

Simple calculation shows that

$$P^N(s) = \frac{P^N(t_N)}{1 + P^N(t_N)(t_N - s)} = \frac{h(t_{N-1})}{1 + h(t_{N-1})(t_N - s)}, \quad s \in [t_{N-1}, t_N]. \quad (A.2)$$

The optimal control trajectory is the solution to the following closed-loop system

$$\begin{cases} \dot{\bar{X}}_N(s) = -P^N(s)\bar{X}_N(s), & s \in [t_{N-1}, t_N], \\ \bar{X}_N(t_{N-1}) = x, \end{cases} \quad (A.3)$$

which is given by

$$\bar{X}_N(s) = x \frac{1 + P^N(t_N)(t_N - s)}{1 + P^N(t_N)(t_N - t_{N-1})} = x \frac{1 + h(t_{N-1})(t_N - s)}{1 + h(t_{N-1})(t_N - t_{N-1})}, \quad s \in [t_{N-1}, t_N], \quad (A.4)$$

and the optimal control is given by

$$\bar{u}_N(s) = -P^N(s)\bar{X}_N(s) = -\frac{xP^N(t_N)}{1 + P^N(t_N)(t_N - t_{N-1})} = -\frac{xh(t_{N-1})}{1 + h(t_{N-1})(t_N - t_{N-1})}, \quad s \in [t, T]. \quad (A.5)$$

Now, on  $[t_{N-2}, t_{N-1}]$ , we consider state equation

$$\begin{cases} \dot{X}_{N-1}(s) = u_{N-1}(s), & s \in [t_{N-2}, t_{N-1}], \\ X_{N-1}(t_{N-2}) = x, \end{cases} \quad (3.31)$$

with the cost functional

$$\begin{aligned}
J_{N-1}(t_{N-2}, x; u_{N-1}(\cdot)) &= \int_{t_{N-2}}^{t_{N-1}} u_{N-1}(s)^2 ds + \int_{t_{N-1}}^{t_N} \bar{u}_N(s)^2 ds + h(t_{N-2})\bar{X}(t_N)^2 \\
&= \int_{t_{N-2}}^{t_{N-1}} u_{N-1}(s)^2 ds + \frac{h(t_{N-1})^2(t_N - t_{N-1}) + h(t_{N-2})}{[1 + h(t_{N-1})(t_N - t_{N-1})]^2} X(t_{N-1})^2 \\
&= \int_{t_{N-2}}^{t_{N-1}} u_{N-1}(s)^2 ds + \frac{P^N(t_N)^2(t_N - t_{N-1}) + h(t_{N-2})}{[1 + P^N(t_N)(t_N - t_{N-1})]^2} X(t_{N-1})^2
\end{aligned}$$

For the corresponding LQ problem, the Riccati equation is

$$\begin{cases} \dot{P}^{N-1}(s) - P^{N-1}(s)^2 = 0, & s \in [t_{N-2}, t_{N-1}], \\ P^{N-1}(t_{N-1}) = \frac{P^N(t_N)^2(t_N - t_{N-1}) + h(t_{N-2})}{[1 + P^N(t_N)(t_N - t_{N-1})]^2}. \end{cases} \quad (A.1)$$

Simple calculation shows that

$$\begin{aligned}
P^{N-1}(s) &= \frac{P^{N-1}(t_{N-1})}{1 + P^{N-1}(t_{N-1})(t_{N-1} - s)} \\
&= \frac{\frac{P^N(t_N)^2(t_N - t_{N-1}) + h(t_{N-2})}{[1 + P^N(t_N)(t_N - t_{N-1})]^2}}{1 + \frac{P^N(t_N)^2(t_N - t_{N-1}) + h(t_{N-2})}{[1 + P^N(t_N)(t_N - t_{N-1})]^2}(t_{N-1} - s)} \\
&= \frac{P^N(t_N)^2(t_N - t_{N-1}) + h(t_{N-2})}{[1 + P^N(t_N)(t_N - t_{N-1})]^2 + [P^N(t_N)^2(t_N - t_{N-1}) + h(t_{N-2})](t_{N-1} - s)}, \\
&= \frac{h(t_{N-1})^2(t_N - t_{N-1}) + h(t_{N-2})}{[1 + h(t_{N-1})(t_N - t_{N-1})]^2 + [h(t_{N-1})^2(t_N - t_{N-1}) + h(t_{N-2})](t_{N-1} - s)}, \\
&\quad s \in [t_{N-2}, t_{N-1}].
\end{aligned} \quad (A.2)$$

Note

$$\begin{aligned}
&P^N(t_{N-1}) - P^{N-1}(t_{N-1}) \\
&= \frac{P^N(t_N)}{1 + P^N(t_N)(t_N - t_{N-1})} - \frac{P^N(t_N)^2(t_N - t_{N-1}) + h(t_{N-2})}{[1 + P^N(t_N)(t_N - t_{N-1})]^2} \\
&= \frac{P^N(t_N)[1 + P^N(t_N)(t_N - t_{N-1})] - P^N(t_N)^2(t_N - t_{N-1}) - h(t_{N-2})}{[1 + P^N(t_N)(t_N - t_{N-1})]^2} \\
&= \frac{P^N(t_N) - h(t_{N-2})}{[1 + P^N(t_N)(t_N - t_{N-1})]^2} = \frac{h(t_{N-1}) - h(t_{N-2})}{[1 + h(t_{N-1})(t_N - t_{N-1})]^2}
\end{aligned}$$

The optimal trajectory is the solution to the following closed-loop system

$$\begin{cases} \dot{\bar{X}}_{N-1}(s) = -P^{N-1}(s)\bar{X}_{N-1}(s), & s \in [t_{N-2}, t_{N-1}], \\ \bar{X}_{N-1}(t_{N-2}) = x, \end{cases} \quad (A.3)$$

which is given by

$$\bar{X}_{N-1}(s) = x \frac{1 + P^{N-1}(t_{N-1})(t_{N-1} - s)}{1 + P^{N-1}(t_{N-1})(t_{N-1} - t_{N-2})}, \quad s \in [t_{N-2}, t_{N-1}], \quad (A.4)$$

and the optimal control is given by

$$\bar{u}_{N-1}(s) = -P^{N-1}(s)\bar{X}_{N-1}(s) = -\frac{xP^{N-1}(t_{N-1})}{1 + P^{N-1}(t_{N-1})(t_{N-1} - t_{N-2})}, \quad s \in [t, T]. \quad (A.5)$$

Now, on  $[t_{N-3}, t_{N-2}]$ , we consider state equation

$$\begin{cases} \dot{X}_{N-2}(s) = u_{N-2}(s), & s \in [t_{N-3}, t_{N-2}], \\ X_{N-2}(t_{N-3}) = x, \end{cases} \quad (3.32)$$

with the cost functional

$$\begin{aligned} & J_{N-2}(t_{N-3}, x; u_{N-2}(\cdot)) \\ &= \int_{t_{N-3}}^{t_{N-2}} u_{N-2}(s)^2 ds + \int_{t_{N-2}}^{t_{N-1}} \bar{u}_{N-1}(s)^2 ds + \int_{t_{N-1}}^{t_N} \bar{u}_N(s)^2 ds + h(t_{N-3})\bar{X}(t_N)^2 \\ &= \int_{t_{N-3}}^{t_{N-2}} u_{N-2}(s)^2 ds + \frac{P^{N-1}(t_{N-1})^2(t_{N-1} - t_{N-2})}{[1 + P^{N-1}(t_{N-1})(t_{N-1} - t_{N-2})]^2} X_{N-2}(t_{N-2})^2 \\ &\quad + \frac{P^N(t_N)^2(t_N - t_{N-1}) + h(t_{N-3})}{[1 + P^N(t_N)(t_N - t_{N-1})]^2} \bar{X}_{N-1}(t_{N-1})^2 \\ &= \int_{t_{N-3}}^{t_{N-2}} u_{N-2}(s)^2 ds + \frac{P^{N-1}(t_{N-1})^2(t_{N-1} - t_{N-2})}{[1 + P^{N-1}(t_{N-1})(t_{N-1} - t_{N-2})]^2} X_{N-2}(t_{N-2})^2 \\ &\quad + \frac{P^N(t_N)^2(t_N - t_{N-1}) + h(t_{N-3})}{[1 + P^N(t_N)(t_N - t_{N-1})]^2 [1 + P^{N-1}(t_{N-1})(t_{N-1} - t_{N-2})]^2} X_{N-2}(t_{N-2})^2 \\ &= \int_{t_{N-2}}^{t_{N-1}} u_{N-1}(s)^2 ds + \frac{P^N(t_N)^2(t_N - t_{N-1}) + h(t_{N-2})}{[1 + P^N(t_N)(t_N - t_{N-1})]^2} X(t_{N-1})^2 \end{aligned}$$

For the corresponding LQ problem, the Riccati equation is

$$\begin{cases} \dot{P}^{N-1}(s) - P^{N-1}(s)^2 = 0, & s \in [t_{N-2}, t_{N-1}], \\ P^{N-1}(t_{N-1}) = \frac{P^N(t_N)^2(t_N - t_{N-1}) + h(t_{N-2})}{[1 + P^N(t_N)(t_N - t_{N-1})]^2}. \end{cases} \quad (A.1)$$

Simple calculation shows that

$$\begin{aligned} P^{N-1}(s) &= \frac{P^{N-1}(t_{N-1})}{1 + P^{N-1}(t_{N-1})(t_{N-1} - s)} \\ &= \frac{\frac{P^N(t_N)^2(t_N - t_{N-1}) + h(t_{N-2})}{[1 + P^N(t_N)(t_N - t_{N-1})]^2}}{1 + \frac{P^N(t_N)^2(t_N - t_{N-1}) + h(t_{N-2})}{[1 + P^N(t_N)(t_N - t_{N-1})]^2} (t_{N-1} - s)} \\ &= \frac{P^N(t_N)^2(t_N - t_{N-1}) + h(t_{N-2})}{[1 + P^N(t_N)(t_N - t_{N-1})]^2 + [P^N(t_N)^2(t_N - t_{N-1}) + h(t_{N-2})](t_{N-1} - s)}, \\ &= \frac{h(t_{N-1})^2(t_N - t_{N-1}) + h(t_{N-2})}{[1 + h(t_{N-1})(t_N - t_{N-1})]^2 + [h(t_{N-1})^2(t_N - t_{N-1}) + h(t_{N-2})](t_{N-1} - s)}, \\ &\quad s \in [t_{N-2}, t_{N-1}]. \end{aligned} \quad (A.2)$$

Note

$$\begin{aligned}
& P^N(t_{N-1}) - P^{N-1}(t_{N-1}) \\
&= \frac{P^N(t_N)}{1 + P^N(t_N)(t_N - t_{N-1})} - \frac{P^N(t_N)^2(t_N - t_{N-1}) + h(t_{N-2})}{[1 + P^N(t_N)(t_N - t_{N-1})]^2} \\
&= \frac{P^N(t_N)[1 + P^N(t_N)(t_N - t_{N-1})] - P^N(t_N)^2(t_N - t_{N-1}) - h(t_{N-2})}{[1 + P^N(t_N)(t_N - t_{N-1})]^2} \\
&= \frac{P^N(t_N) - h(t_{N-2})}{[1 + P^N(t_N)(t_N - t_{N-1})]^2} = \frac{h(t_{N-1}) - h(t_{N-2})}{[1 + h(t_{N-1})(t_N - t_{N-1})]^2}
\end{aligned}$$

The optimal trajectory is the solution to the following closed-loop system

$$\begin{cases} \dot{\bar{X}}_{N-1}(s) = -P^{N-1}(s)\bar{X}_{N-1}(s), & s \in [t_{N-2}, t_{N-1}], \\ \bar{X}_{N-1}(t_{N-2}) = x, \end{cases} \quad (A.3)$$

which is given by

$$\bar{X}_{N-1}(s) = x \frac{1 + P^{N-1}(t_{N-1})(t_{N-1} - s)}{1 + P^{N-1}(t_{N-1})(t_{N-1} - t_{N-2})}, \quad s \in [t_{N-2}, t_{N-1}], \quad (A.4)$$

and the optimal control is given by

$$\bar{u}_{N-1}(s) = -P^{N-1}(s)\bar{X}_{N-1}(s) = -\frac{xP^{N-1}(t_{N-1})}{1 + P^{N-1}(t_{N-1})(t_N - t_{N-1})}, \quad s \in [t, T]. \quad (A.5)$$

Now, if we let

$$J(t; \tau, y; u(\cdot)) = \int_{\tau}^T u(s)^2 ds + h(t)X(T; \tau, y, u(\cdot))^2, \quad \tau \in [t, T], \quad (A.6)$$

then the optimal value function (for fixed  $t$ ) is given by

$$\begin{aligned} V(t; \tau, y) &\equiv \inf_{u(\cdot) \in \mathcal{U}[\tau, T]} J(t; \tau, y; u(\cdot)) = J(t; \tau, y; \bar{u}(\cdot)) = P(\tau; t)y^2 \\ &= \frac{h(t)}{1 + h(t)(T - \tau)} y^2, \quad \forall (\tau, y) \in [t, T] \times \mathbb{R}. \end{aligned} \quad (A.7)$$

Next, let  $\tau \in (t, T)$ , we consider Problem (C) on  $[\tau, T]$  with initial state

$$y = \bar{X}(\tau; t, x) = x \frac{1 + h(t)(T - \tau)}{1 + h(t)(T - t)}. \quad (A.8)$$

The same as above, we see that the corresponding solution to the Riccati equation is given by

$$P(s; \tau) = \frac{h(\tau)}{1 + h(\tau)(T - s)}, \quad s \in [\tau, T], \quad (A.9)$$

and

$$\begin{aligned} \inf_{u(\cdot) \in \mathcal{U}[\tau, T]} J(\tau; \tau, y; u(\cdot)) &= P(\tau; \tau)y^2 = \frac{h(\tau)y^2}{1 + h(\tau)(T - \tau)} \\ &= \frac{x^2 h(\tau)[1 + h(t)(T - \tau)]^2}{[1 + h(\tau)(T - \tau)][1 + h(t)(T - t)]^2}. \end{aligned} \quad (A.10)$$

However,

$$\begin{aligned} J(\tau, y; \bar{u}(\cdot)) &= \int_{\tau}^T \bar{u}(s)^2 ds + h(\tau)X(T; \tau, y, \bar{u}(\cdot))^2 \\ &= \frac{x^2 h(t)^2 (T - \tau)}{[1 + h(t)(T - t)]^2} + h(\tau) \left[ y - \frac{x h(t)(T - \tau)}{1 + h(t)(T - t)} \right]^2 \\ &= \frac{x^2 h(t)^2 (T - \tau)}{[1 + h(t)(T - t)]^2} + \frac{x^2 h(\tau)}{[1 + h(t)(T - t)]^2}. \end{aligned} \quad (A.11)$$

Hence,

$$\begin{aligned} &J(\tau, y; \bar{u}(\cdot)) - \inf_{u(\cdot) \in \mathcal{U}[\tau, T]} J(\tau, y; u(\cdot)) \\ &= \frac{x^2 h(t)^2 (T - \tau) + x^2 h(\tau)}{[1 + h(t)(T - t)]^2} - \frac{x^2 h(\tau)[1 + h(t)(T - \tau)]^2}{[1 + h(\tau)(T - \tau)][1 + h(t)(T - t)]^2} \\ &= x^2 \left[ \frac{[h(t)^2 (T - \tau) + h(\tau)][1 + h(\tau)(T - \tau)] - h(\tau)[1 + h(t)(T - \tau)]^2}{[1 + h(\tau)(T - \tau)][1 + h(t)(T - t)]^2} \right] \\ &= \frac{x^2 [h(\tau) - h(t)]^2 (T - \tau)}{[1 + h(\tau)(T - t)][1 + h(t)(T - \tau)]^2} > 0, \quad \text{unless } h(\tau) = h(t) \text{ or } x = 0. \end{aligned} \quad (A.12)$$

This shows that the restriction of  $\bar{u}(\cdot; t, x)$  on  $[\tau, T]$  is not necessarily optimal for Problem (C) with initial pair  $(\tau, \bar{X}(\tau; t, x))$ . Hence, Problem (C) is time-inconsistent.